

*Discontinuous Fluid Motion past a Bent Plane, with Special
Reference to Aeroplane Problems.*

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The following investigation is based on the Kirchhoff-Helmholtz theory of discontinuous stream-line motion of a fluid moving through an orifice or past a barrier. The problem is two-dimensional and the fluid is assumed to be frictionless and incompressible.

The question has been discussed fully by Sir G. Greenhill, F.R.S., in his report to the Advisory Committee for Aeronautics, and in his book on the Dynamics of Mechanical Flight. He considers at length the fluid motion past a barrier, but in what follows a bent barrier will be considered.

Greenhill refers to a problem similar to this, where the stream is supposed to divide at the bend. The present case can be reduced to that one, by making two parametric angles equal ($P = B$).

Integrals are obtained giving the lengths of and thrusts on the planes, and by means of approximate methods these will be evaluated. It will also be shown how these integrals can be evaluated exactly, but the expressions to which they have been reduced are so long that they cannot be of great value in investigating special cases.

The results arrived at will be used in finding the lift and drift on the surface, and the relative advantages of planes bent at different angles and along different lines (parallel to the edges) will be considered.

Supposing the surface to be the main plane of an aeroplane, and the fluid to be air, the numerical results cannot be expected to be strictly quantitatively correct, inasmuch as the motion is supposed two-dimensional and the fluid here considered is assumed to be frictionless and incompressible, which is by no means true for actual fluids. Consequently, instead of a region of dead water, separated by a surface of discontinuity from the moving part, eddies are set up in the wake of a surface moving through air.

Qualitative conclusions can, however, be drawn, and the effect of camber compared for various planes, and in this respect our results appear to agree with experimental results.

§ 1. *List of the Functions employed.*

Although the theory of discontinuous motion is well known and explained, not always very lucidly, in text-books, it will be convenient to briefly summarise the functions and transformations employed, as follows:—

$z = x + iy$; x and y are the co-ordinates of a point in the z diagram, being a representation of the cross section of the planes and the fluid.

$w = \phi + i\psi$; ϕ velocity potential, ψ stream function. In the w diagram lines parallel to the real axis represent the stream-lines.

$$\begin{aligned}\zeta &= Q(dz/dw), \text{ where } Q \text{ is the velocity at infinity,} \\ \text{mod } \zeta &= Q/q, \text{ where } q \text{ is the resultant velocity at any point,} \\ \arg \zeta &= \theta, \text{ the directional angle of this velocity,} \\ \Omega &= \log \zeta = \log (Q/q) + i\theta.\end{aligned}$$

In the Ω diagram fixed straight boundaries are represented by the lines $\theta = \text{constant}$ parallel to the real axis, and free boundaries of surfaces of discontinuity by straight lines parallel to the imaginary axis (since by the pressure equation q is constant) usually by the imaginary axis itself $\log (Q/q) = 0$ corresponding to $q = Q$.

In the case of a stream of infinite breadth impinging on a single barrier, and dividing, at a point P in the barrier, into parts flowing past the two sides, the dividing stream line in the w diagram invariably takes the form shown in fig. 3 (§ 2), and it is necessary to apply the transformation $t = \sqrt{w}$ in order to reduce the diagram to the form of fig. 4 (§ 2), the angle between PD and PE thus being changed from 360° to 180° .

This transformation is, of course, a case of the Schwarz-Christoffel transformation

$$d\Omega/dt = \text{const.} (t-a)^{\alpha/\pi-1} (t-b)^{\beta/\pi-1} (t-c)^{\gamma/\pi-1} \dots,*$$

by means of which points on the sides AB, BC, ... , A of a closed rectilinear polygon in the Ω diagram are represented by points on the real axis in the t diagram; a, b, c , etc., corresponding to the corners A, B, C, etc., of the polygon at which the internal angles are α, β, γ , etc. Points within the polygon are represented by points in the positive half plane in t .

As t increases through the value a say the corresponding factor of the modulus changes from $(a-t)^{\alpha/\pi-1}$ to $(t-a)^{\alpha/\pi-1}$, i.e. a factor $(-1)^{\alpha/\pi-1}$ is added to $d\Omega/dt$, so that the argument in the Ω diagram changes by $\pi - \alpha$.

The formula can of course be written

$$d\Omega/dt = \text{const.} (t-a)^{\alpha'/\pi} (t-b)^{\beta'/\pi} \dots,$$

α', β' , etc., being the external angles corresponding to α, β , etc.

If we assume then that the relations connecting w and t , and Ω and t , are known the problem is solved.

$$\begin{aligned}\text{Let} \quad dw/dt &= F(t), & d\Omega/dt &= G(t), \\ \Omega &= \log \zeta = \log (dz/dw), & \text{when } Q &= 1.\end{aligned}$$

* Forsyth, 'Theory of Functions,' § 267 (1900).

But

$$\Omega = \int G(t) dt.$$

Therefore

$$\zeta = e^{\int G(t) dt}, \quad \text{i.e.} \quad dz/dw = e^{\int G(t) dt}.$$

Therefore

$$dz/dt = e^{\int G(t) dt} F(t),$$

which gives a relation between the length of the plane and t .

For the thrust we have $\int p \left| \frac{dz}{dt} \right| dt$, taken over the front and back of the plane, so that if the density of the fluid be ρ , the thrust is

$$\frac{\rho}{2} \int (1 - q^2) \left| \frac{dz}{dt} \right| dt, \quad \text{when } Q = 1.$$

§ 2. *The Theory of Discontinuous Motion Applied to a Bent Plane.*

After having given a list of the functions employed, we proceed to use them to find the thrust on a bent lamina.

Let ABC be a section of the lamina, the relative motion of the fluid being denoted by the arrows (fig. 1). Let α and β be the angles AB and BC respectively make with the direction of the fluid at infinity. Let P be the point at which the stream divides.

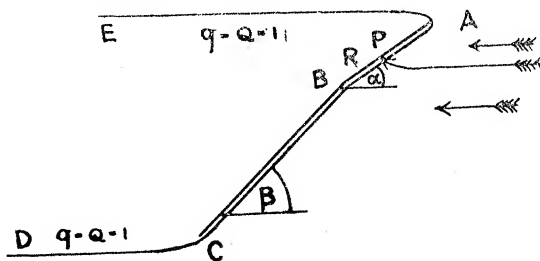


FIG. 1.

The velocity vanishes at P. The velocity also vanishes at the bend B,* and must therefore have a maximum at some point R in BP.

Let $Q = 1$ (until § 6), then

$$\Omega = \log(1/q) + i\theta.$$

If $\theta = \alpha$ along PB, $\theta = \beta$ along BC and $\alpha - \pi$ along PA,

At D and E $q = 1$ and $\theta = 0$, and $q = 0$ at P and B.

Consequently the Ω diagram is as shown in fig. 2.

* Lamb, 'Hydrodynamics' § 63 (1906). The angle ABC (above, fig. 1) $< \pi$.

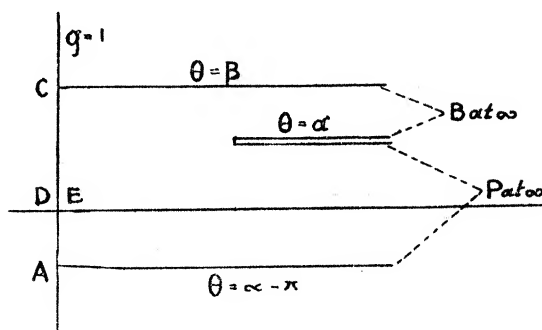


FIG. 2.

$w = \phi + i\psi$ is represented in fig. 3, and we assume the integration constants so chosen that $w = 0$ at P.

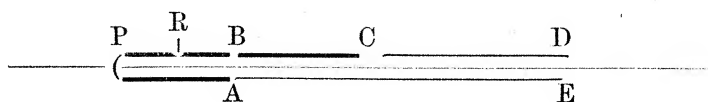


FIG. 3.

Finally, with $t-p = \sqrt{w}$, we obtain the t diagram, as given in fig. 4.



FIG. 4.

The Schwarz-Christoffel formula is now applied to transform the Ω diagram to the t diagram. In this case it is

$$\frac{d\Omega}{dt} = \frac{\text{const.}(t-r)}{(t-b)(t-p)\sqrt{(t-a)(t-c)}},$$

where $t = r, b, p, a$, and c are the values of t at R, B, P, A, and C respectively.

Splitting up by partial fractions and putting $y = 1/(t-b)$ in one case, and $= 1/(t-p)$ in the other, and integrating we have

$$\begin{aligned} \Omega = A_1 \log \frac{\sqrt{(t-a)(c-b)} + \sqrt{(t-c)(a-b)}}{\sqrt{(t-b)(c-a)}} \\ + A_2 \log \frac{\sqrt{(t-a)(c-p)} + \sqrt{(t-c)(a-p)}}{\sqrt{(t-p)(c-a)}}. \end{aligned} \quad (1)$$

Referring now to the Ω and t diagrams, figs. 2 and 4, it is evident that as t increases through P, Ω increases through π , and as t increases through B,

Ω increases through $\beta - \alpha$. Now, calling $\beta - \alpha = \pi/2\nu$ and neglecting the constant factor which occurs throughout, we may write

$$\Omega = 2 \log \frac{\sqrt{(t-a)(c-b)} + \sqrt{(t-c)(a-b)}}{\sqrt{(t-b)(c-a)}} + \frac{1}{\nu} \log \frac{\sqrt{(t-a)(c-b)} + \sqrt{(t-c)(a-b)}}{\sqrt{(t-b)(c-a)}}. \quad (2)$$

Let us now choose the scale of representation so that $c - a = 1$. Then for $c > t > a$ there is a real angle U between 0 and $\frac{1}{2}\pi$, such that $c - t = \cos^2 U$ and $t - a = \sin^2 U$.*

This gives us

$$\text{i. } w = (t - p)^2 = (\sin^2 U - \sin^2 P)^2 = \sin^2(U + P) \sin^2(U - P). \quad (3)$$

$$\text{ii. } \sqrt{(t-a)(c-b)} + \sqrt{(t-c)(a-b)} = \sin(U + B), \text{ and so on.}$$

Consequently,

$$\Omega = \log \frac{\sin(U + P)}{\sin(U - P)} + \frac{1}{2\nu} \log \frac{\sin(U + B)}{\sin(U - B)}. \quad (4)$$

The Angles of Attack.—By the term the “angles of attack” we understand the angles the two parts of the bent plane make with the direction of the stream at infinity.

Any particular plane has the following characteristics: P , B , and ν . B depends on the ratio of AB to BC (fig. 1) and ν on the angle between AB and BC . We now proceed to find an expression for P by considering the angles of attack.

AB will be henceforth called the front plane, and BC the rear plane.

Now

$$\Omega = A_1 \log \frac{\sqrt{(t-a)(c-b)} + \sqrt{(t-c)(a-b)}}{\sqrt{(t-b)(c-a)}} + A_2 \log \frac{\sqrt{(t-a)(c-p)} + \sqrt{(t-c)(a-p)}}{\sqrt{(t-p)(c-a)}} = \log \frac{1}{q} + i\theta.$$

First of all, when $c > t > b$ it is evident that, with $c > b > p > a$, the right-hand side of the above equation is real, therefore

$$\theta = 0.$$

This is along the rear plane.

Again, when $b > t > p$, the part whose coefficient is A_2 is real, whereas the other is imaginary and $= A_1 \log(1/i)$.

$$\text{Therefore} \quad \theta = A_1 \log(1/i) = -A_1 \pi/2.$$

This is along PB . Finally, $p > t > a$ gives us

$$\theta = (A_1 + A_2) \log(1/i) = -(A_1 + A_2) \pi/2$$

along AP .

* Cf. Greenhill ‘The Dynamics of Mechanical Flight.’

From fig. 5 we see that if $\theta = 0$ along BC, the values of θ along BP and AP are $-\pi/2\nu$ and $-(\pi + \pi/2\nu)$ respectively, giving

$$A_1 = 1/\nu \quad \text{and} \quad A_2 = 2,$$

as before.

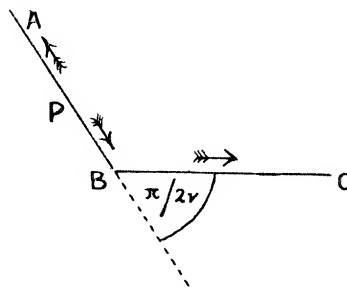


FIG. 5.

Again, for $t = \infty$,

$$\begin{aligned} \Omega &= A_1 \log \frac{\sqrt{(c-b)} + \sqrt{(a-b)}}{\sqrt{(c-a)}} + A_2 \log \frac{\sqrt{(c-p)} + \sqrt{(a-p)}}{\sqrt{(c-a)}} \\ &= A_1 \log (\cos B + i \sin B) + A_2 \log (\cos P + i \sin P) \end{aligned}$$

Hence (5)
$$\theta = 2P + B/\nu.$$

This, then, is the angle of attack on the rear plane.

We will confine ourselves to positive angles of attack. The angle of attack on the front plane is

$$2P + B/\nu - \pi/2\nu. \tag{6}$$

This is by assumption positive.

Hence (7)
$$2P > \pi/2\nu - B/\nu.$$

Hence the stream must divide beyond a certain point defined by

$$2P = \frac{1}{2}\pi - B/\nu,$$

which is the limiting position in the case of the front plane being placed parallel to the direction of the stream at infinity.

The Lengths of the Planes.

The lengths of the planes are given by $\int_a^b \left| \frac{dz}{dt} \right| dt$ and $\int_b^c \left| \frac{dz}{dt} \right| dt$. Considering the latter, it can be written in the form $\int_B^{\frac{1}{2}\pi} \left| \frac{dz}{dU} \right| dU$, or, since $\theta = 0$ along the rear plane and z real, in the form $\int_{z_B}^{z_{\frac{1}{2}\pi}} dz$, z_B and $z_{\frac{1}{2}\pi}$ denoting the values of z when $U = B$ and $\frac{1}{2}\pi$ respectively.

Now
$$\zeta = dz/dv, \quad \text{and} \quad w = (\sin^2 U - \sin^2 P)^2;$$

therefore

$$L_2 \text{ (the length of rear plane)} = 4 \int_B^{\frac{1}{2}\pi} \zeta \sin U \cos U \sin(U+P) \sin(U-P) dU.$$

Substituting for ζ from (4) $\log \zeta = \Omega$, it is immediately reduced to the form

$$L_2 = 2 \int_B^{\frac{1}{2}\pi} \sin^2(U+P) \sin 2U \left\{ \frac{\sin(U+B)}{\sin(U-B)} \right\}^{1/2\nu} dU. \quad (8)$$

$$\text{Similarly} \quad L_1 = 2 \int_0^B \sin^2(U+P) \sin 2U \left\{ \frac{\sin(B+U)}{\sin(B-U)} \right\}^{1/2\nu} dU, \quad (9)$$

for the front plane.

The Pressures on the Planes.

The pressures are given by

$$\frac{\rho}{2} \int_a^b (1-q^2) \left| \frac{dz}{dt} \right| dt \text{ and } \frac{\rho}{2} \int_b^c (1-q^2) \left| \frac{dz}{dt} \right| dt. \quad \rho = \text{density of fluid.}$$

We first of all proceed to find $\int q^2 \left| \frac{dz}{dt} \right| dt$ for the rear plane. This and the corresponding integrals will be called the Pressure Difference Integrals.

$\theta = 0$, as before, along the rear plane, hence we write

$$P_2 = \int_{z_B}^{z_{\frac{1}{2}\pi}} q^2 dz = \int \left(\frac{dw}{dz} \right)^2 dz = \int \frac{dw}{\zeta}.$$

Proceeding as in the case of the lengths, we obtain

$$P_2 = 2 \int_B^{\frac{1}{2}\pi} \sin^2(U-P) \sin 2U \left\{ \frac{\sin(U-B)}{\sin(U+B)} \right\}^{1/2\nu} dU, \quad (10)$$

and, similarly,

$$P_1 = 2 \int_0^B \sin^2(U-P) \sin 2U \left\{ \frac{\sin(B-U)}{\sin(B+U)} \right\}^{1/2\nu} dU. \quad (11)$$

§ 3. *A Further Application to the Case of a Plane with more than One Bend.*

Suppose now that we have a plane bent at several points $A_2, A_3, \dots, A_r, \dots, A_{n-1}$. Let the angles the parts $A_1A_2, A_2A_r, A_rA_{r+1}, \dots, A_{n-1}A_n$, make with the direction of motion of the fluid at infinity be $\alpha_2, \alpha_r, \alpha_{r+1}, \dots, \alpha_n$ respectively.

Let the point P where the stream divides be situated between A_r and A_{r+1} , and the points of maximum velocities be $R_2, \dots, R_r, R'_r, R_{r+1}, \dots$, respectively.

A_1E and A_nD are the free stream-lines along which $q = 1$ as before

The z, Ω, w , and t diagrams are as shown in figs. 6, 7, 8, and 9.

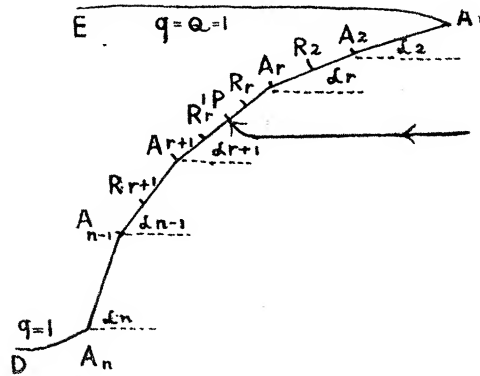


FIG. 6.

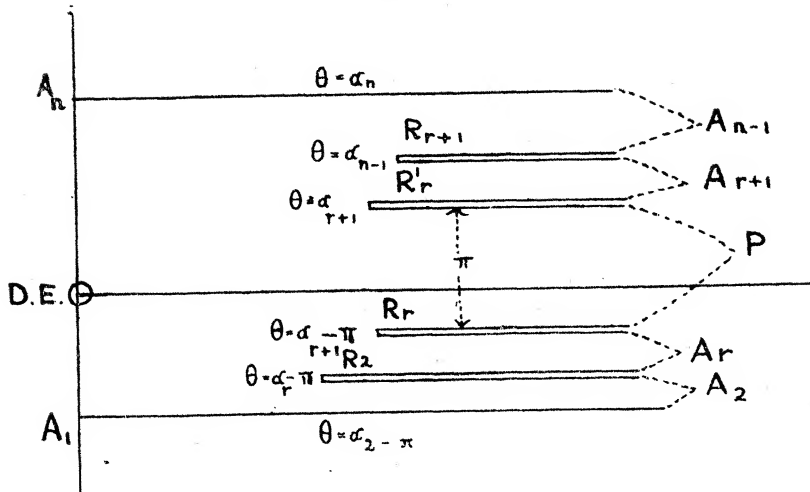


FIG. 7.

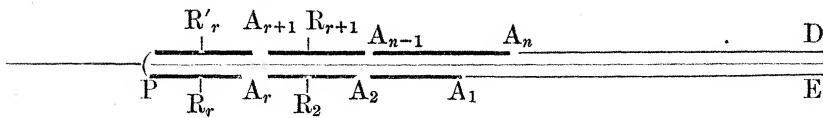


FIG. 8.

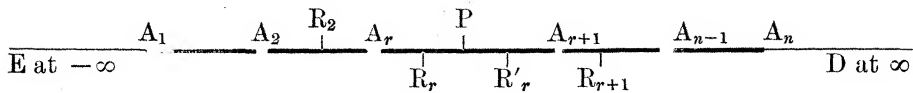


FIG. 9.

We now apply the Schwarz-Christoffel formula to transform fig. 7 to fig. 9; in this case it takes the form

$$\frac{d\Omega}{dt} = \frac{\text{const.} (t-\rho_2)(t-\rho_r)(t-\rho'_r)(t-\rho_{r+1})}{(t-a_2)(t-a_r)(t-p)(t-a_{r+1})(t-a_{n-1})\sqrt{(t-a_1)(t-a_n)}}.$$

$\rho_2, \rho_r, \rho'_r, \rho_{r+1}$ correspond to the values of t at R_2, R_r, R'_r, R_{r+1} , and the a 's to the values t corresponding to the A 's.

For more bends in the plane additional factors would be inserted in the numerator and denominator of this expression.

Now, putting $\alpha_3 - \alpha_2 = \pi/2\nu_2$, $\alpha_{r+1} - \alpha_r = \pi/2\nu_r$, etc., we have, on splitting up the right-hand side of this equation by means of partial fractions and determining the constants as in the previous case,

$$\begin{aligned} \frac{d\Omega}{dt} = & \frac{\sqrt{(a_p - a_r)(a_p - a_1)}}{(t - a_p)\sqrt{(t - a_n)(t - a_1)}} + \frac{1}{2\nu_2} \cdot \frac{\sqrt{(a_2 - a_n)(a_2 - a_1)}}{(t - a_2)\sqrt{(t - a_n)(t - a_1)}} + \dots \\ & + \frac{1}{2\nu_r} \cdot \frac{\sqrt{(a_r - a_n)(a_r - a_1)}}{(t - a_r)\sqrt{(t - a_r)(t - a_1)}} + \frac{1}{2\nu_{r+1}} \cdot \frac{\sqrt{(a_{r+1} - a_n)(a_{r+1} - a_1)}}{(t - a_{r+1})\sqrt{(t - a_n)(t - a_1)}} + \dots \\ & + \frac{1}{2\nu_{n-1}} \cdot \frac{\sqrt{(a_{n-1} - a_n)(a_{n-1} - a_1)}}{(t - a_{n-1})\sqrt{(t - a_n)(t - a_1)}}. \end{aligned} \quad (12)$$

Integrating as before, and remembering that $\zeta = e^\Omega$, we have

$$\zeta = \left\{ \frac{\sqrt{(a_n - a_p)(t - a_1)} + \sqrt{(a_1 - a_p)(t - a_n)}}{\sqrt{(a_n - a_1)(t - a_p)}} \right\}^2 + \prod_{r=2}^{r=n-1} \left\{ \frac{\sqrt{(a_n - a_r)(t - a_1)} + \sqrt{(a_1 - a_r)(t - a_n)}}{\sqrt{(a_n - a_1)(t - a_r)}} \right\}^{1/\nu_r}.$$

To find the angle of attack on the part $A_{n+1}A_n$ (fig. 6), put $t = \infty$ in the equation for Ω (not given here, but obtained from (12) by integration).

$$\Omega = 2 \log \frac{\sqrt{(a_n - a_p)} + \sqrt{(a_1 - a_p)}}{\sqrt{(a_n - a_1)}} + \sum_{r=2}^{r=n-1} \frac{1}{\nu_r} \log \frac{\sqrt{(a_n - a_r)} + \sqrt{(a_1 - a_r)}}{\sqrt{a_{n-1}}}.$$

Now, $a_n > a_{n-1} > \dots > a_{r+1} > a_r > \dots > a_1$, hence the imaginary part of this expression

$$= i \left\{ 2 \tan^{-1} \sqrt{\left(\frac{a_p - a_1}{a_n - a_p} \right)} + \sum_{r=2}^{r=n-1} \frac{1}{\nu_r} \tan^{-1} \sqrt{\left(\frac{a_r - a_1}{a_n - a_r} \right)} \right\}.$$

or using the same artifice as before (3, § 2)

$$= i \left(2P + \sum_{r=2}^{r=n-1} \frac{1}{\nu_r} U_r \right),$$

where U_r is the value of U corresponding to a_r .

Therefore $\theta = 2P + \sum_{r=2}^{r=n-1} \frac{1}{\nu_r} U_r$ (compare 5, § 2).

θ is known from the conditions of motion about any particular plane, $U_2, \dots, U_{n-1}$ are also known (depending on the points at which the section of the plane is bent); hence P can be determined.

§ 4. *Reduction of the Integrals.*

When $1/2\nu$ is an exact fraction l/n say, the integrals are reducible to finite form.

Taking, for example, the integral

$$\int_B^{\frac{1}{2}\pi} \sin^2(U+P) \sin 2U \left\{ \frac{\sin(U+B)}{\sin(U-B)} \right\}^{1/2\nu} dU,$$

and writing $\frac{\sin(U+B)}{\sin(U-B)} = z^n,$

it will be seen, by expanding the first two factors of the integrand in powers of $\sin U$ and $\cos U$, that the integral can be expressed as the sum of integrals of the form

$$\int \sin^p U \cos^q U z^l dU, \quad \text{where } p+q = 4.$$

If we now change the variable from U to z , we have

$$\begin{aligned} \frac{\tan U}{\tan B} &= \frac{z^n + 1}{z^n - 1}, & dU &= \frac{-nz^{n-1} \sin 2B dz}{z^{2n} - 2z^n \cos 2B + 1}, \\ \cos U &= \frac{(z^n - 1) \cos B}{\sqrt{(z^{2n} - 2z^n \cos 2B + 1)}}, & \text{and } \sin U &= \frac{(z^n + 1) \sin B}{\sqrt{(z^{2n} - 2z^n \cos 2B + 1)}}. \end{aligned}$$

The integrals are all thus reduced to those of rational fractions having as denominator $(z^{2n} - 2z^n \cos 2B + 1)^3$ and as numerator integral powers of z , and these can easily be integrated by means of partial fractions. While the functions involved have been shown to be "integrable," the results are obviously far too cumbersome to be suitable for the purpose of calculation, and we therefore have recourse to approximate methods of integration.

§ 5. *Approximate Evaluation of Integrals.*

"Simpson's Rules" will be used to evaluate the integrals where they are applicable, but it is evident that in the neighbourhood of $U = B$, they fail. The following method will consequently be used near $U = B$.

Consider $\int \sin^2(U-P) \sin 2U \left\{ \frac{\sin(U-B)}{\sin(U+B)} \right\}^m dU.$

Put $U-B = x$, and the integrand becomes

$$\sin^m x \frac{\sin^2(x+B-P) \sin 2(x+B)}{\sin^m(x+2B)} = F(x), \text{ say.}$$

When $x \rightarrow 0$ (i.e. $U \rightarrow B$), $\sin^m x \rightarrow x^m$, and we may write $F(x) = x^m \phi(x)$.

$\phi(x)$ is now expanded in powers of x , being finite and continuous within the limits, and $F(x) = (Ax^m + Bx^{m+1} + Cx^{m+2} + \dots) = y$.

Let y_1 be the value of y [$= F(x)$] when $x = h$,

$$\begin{array}{ccc} \vdots & & \\ y_r & \text{,,} & x = rh. \end{array}$$

$$\begin{aligned} \text{Suppose } \int_0^{rh} y dx \text{ (to be required)} &= (P_1 y_1 + \dots + P_r y_r) h \\ &= (P_1 + 2^m P_2 + \dots + r^m P_r) A h^{m+1} + (P_1 + 2^{m+1} P_2 + \dots + r^{m+1} P_r) B h^{m+2} + \dots \end{aligned}$$

$$\text{But } \int_0^{rh} y dx = A (rh)^{m+1}/(m+1) + B (rh)^{m+2}/(m+2) + \dots$$

$$\text{Therefore } P_1 + 2^m P_2 + 3^m P_3 + \dots + r^m P_r = r^{m+1}/(m+1),$$

$$P_1 + 2^{m+1} P_2 + 3^{m+1} P_3 + \dots + r^{m+1} P_r = r^{m+2}/(m+2), \text{ and so on.}$$

which, when solved, give the values of P.

In the cases that will be given the lengths and pressures have been calculated for values of $m = 1/12, 1/18, 1/36, 1/180$, and 0, corresponding to angles $\pi/2\nu = 15^\circ, 10^\circ, 5^\circ, 1^\circ$, and 0° between the planes, and r has been taken = 2 giving then

$$P_1 = 2^{m+2}/(m+1)(m+2),$$

$$P_2 = 2m/(m+1)(m+2).$$

The formula here derived, then, will be used in the immediate vicinity of B, from B to B₁ say, on the front plane; $0 < B_1 < B$; and from B to B₂ on the rear plane, $B < B_2 < \frac{1}{2}\pi$. One of Simpson's Rules will be used for the evaluation of the integrals between B₁ and 0, and B₂ and $\frac{1}{2}\pi$, the intervals being 5° .

To test the accuracy of this procedure, the integral $\int_0^a \frac{dx}{\sqrt{(a^2 - x^2)}}$ has been evaluated and found to be 1.55, as compared with the true value $\frac{1}{2}\pi = 1.57$. The error is, therefore, about 1.5 per cent. with these intervals.

In applying our rules to the integrals, there are two ways of proceeding.

(a) Certain fixed values can be given to P and B, P having, of course, to satisfy the condition assigned to it in (7), § 2, and the total lift and drift with the angles of attack determined by these values of P and B found.

(b) A fixed value can be assigned to B which will determine the ratio of the length of the front to the rear plane, and we have three integrals

$$\int \sin 2U \cdot Z^m dU, \quad \int \sin^2 2U \cdot Z^m dU, \quad \int \sin 4U \cdot Z^m dU,$$

to evaluate, with the coefficients $\frac{1}{2}$, $\frac{1}{2} \sin 2P$, and $-\frac{1}{4} \cos 2P$ respectively, for half the lengths of the planes. For $\sin^2(U+P) \sin 2U$ can be written in the form

$$\frac{1}{2} \sin 2U + \frac{1}{2} \sin 2P \cdot \sin^2 2U - \frac{1}{4} \cos 2P \cdot \sin 4U.$$

[In the pressure difference integrals Z is different, and the sign of P is changed.]

This method has the advantage that after the integrals have been evaluated

we can find the thrusts for any angle of attack, which will determine for us the value of P . One or two cases will be dealt with according to the first method, but the second way will in general, in what follows, be used.

Now, after the lengths of the planes for various values of B have been calculated, they will be reduced in such a way as to make the sum of the semi-lengths equal to $10c$, where c is the circular measure of 5° .

The corresponding lifts and drifts for various angles of attack, and values of m , will accordingly be calculated. Further the thrust on the chord AC (fig. 1, § 2) will be calculated, and results compared.

Finally the lifts and drifts on planes, for different values of B , and the least integral value of P satisfying the condition already cited, will be calculated. This will be done for one value only of m , this being sufficient to give a general conception of the relative advantages of the planes considered.

Calculations have been made for the following values of B ; 35° , 40° , 45° , 60° , and 75° , and for the values of m already given. For any other values of B and m the corresponding results can be found by interpolation.

The thrust on the chord is easily obtained as follows:—

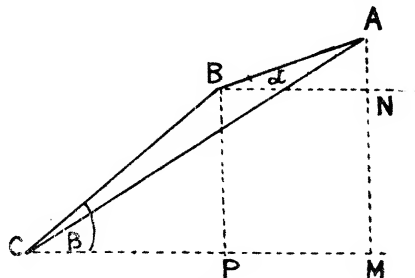


FIG. 10.

BN and CM are parallel to the direction of motion, ANM and BP perpendicular from A and B (fig. 10).

The angles of attack ABN and BCM are α and β respectively.

Let $AB = l_1$ and $BC = l_2$ (which are supposed to be already calculated).

$$AM = AN + NM = l_1 \sin \alpha + l_2 \sin \beta,$$

$$CM = CP + PM = l_1 \cos \alpha + l_2 \cos \beta.$$

Therefore the angle of attack ($\zeta = \angle ACM$) on the chord is

$$\tan^{-1} \frac{l_1 \sin \alpha + l_2 \sin \beta}{l_1 \cos \alpha + l_2 \cos \beta}.$$

Knowing this, P can be found, and then by means of the method (*b*) above with $m = 0$ the thrust can be found.

§ 6. *Numerical Results.*

The following Tables are given, to illustrate the way in which the general results were obtained.

In the first case (Table I)—

$$\begin{aligned}
 B &= 45^\circ. & P &= 5^\circ. & c &= \text{circular measure of } 5^\circ. \\
 \text{Length of front plane, } & 2.78 \times 2c. & \rho &= \text{density of fluid.} \\
 \text{,, rear ,, , } & 7.22 \times 2c. & \alpha &= \text{angle of attack on front plane.} \\
 & & \beta &= \text{,, ,, rear ,,} \\
 & & Q &= \text{velocity of fluid at infinity.}
 \end{aligned}$$

Table I.

<i>m.</i>	$\beta - \alpha.$	Front.			Rear.			Sum.		Chord.	
		$\alpha.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$	$\beta.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$
$\circ$	$\circ$	$\circ$			$\circ$						
$\frac{1}{12}$	15	$2\frac{1}{2}$	1.638	0.068	$17\frac{1}{2}$	2.230	0.702	3.868	0.770	2.976	0.698
$\frac{1}{8}$	10	5	1.361	0.118	15	1.912	0.512	3.273	0.630	2.778	0.554
$\frac{1}{6}$	5	$7\frac{1}{2}$	1.351	0.177	$12\frac{1}{2}$	1.568	0.346	2.919	0.523	2.678	0.530
$\frac{1}{3}$	1	$9\frac{1}{2}$	1.216	0.202	$10\frac{1}{2}$	1.268	0.233	2.484	0.435	2.410	0.432
$\frac{1}{2}$	0	10	1.188	0.208	10	1.188	0.208	2.376	0.416	2.380	0.414

The ratio of the lengths of the two planes has been found in all cases to be approximately independent of m and P . It does not deviate more than 1.5 per cent. from the mean value.

The next Table (II) gives the total lift and drift on a plane in which $B = 75^\circ$ for the cases (1) $P = 2^\circ$, and (2) when the angle of attack on the front plane is zero. The lift and drift on the chord are also given.

The lengths of the front and rear planes in this case are $8.77 \times 2c$ and $1.23 \times 2c$ respectively.

Table II.

$\beta - \alpha.$	Bent plane $P = 2^\circ.$		Bent plane. Front attack $0^\circ.$		Chord of plane $P = 2^\circ.$	
	$L/\rho c Q^2.$	$D/\rho c Q^2.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$	$L/\rho c Q^2.$	$D/\rho c Q^2.$
$\circ$						
15	2.434	0.150	2.111	0.084	0.953	0.057
10	1.918	0.113	1.388	0.036	0.946	0.058
5	1.479	0.092	0.687	0.010	0.958	0.062
1	1.228	0.080	0.189	Negligible	1.006	0.069
0	1.020	0.070	0	0	1.020	0.070

Thirdly, the lift and drift have been calculated for the least integral value of P satisfying the conditions of the problem, and the results given in Table III $m = 1/36$, $\beta - \alpha = 5^\circ$.

Table III.

B.	P.	$\frac{1}{2c}$ length.		$\frac{1}{\rho c Q^2}$ lift.		$\frac{1}{\rho c Q^2}$ drift.		$\frac{1}{\rho c Q^2}$ sum of	
		Front.	Rear.	Front.	Rear.	Front.	Rear.	Lift.	Drift.
°	°								
35	6	1·28	8·72	0·769	2·088	0·094	0·443	2·857	0·537
40	5	2·01	7·99	0·923	1·833	0·116	0·396	2·756	0·412
45	4	2·78	7·22	1·157	1·394	0·111	0·286	2·551	0·367
60	3	5·84	4·26	1·655	0·487	0·125	0·088	2·132	0·213
75	2	8·77	1·23	1·340	0·139	0·073	0·019	1·479	0·092

Finally in Table IV are given the total lift and drift, and the ratio of lift to drift, for a plane whose angle of attack on its front part is 5° , for values of $B = 35^\circ, 45^\circ, 60^\circ$, and 75° , and $m = 1/12, 1/18, 1/36, 1/180$, and 0.

Table IV.

B.	35°.			45°.			60°.			75°.		
$\beta - \alpha$.	$L/\rho c Q^2$.	$D/\rho c Q^2$.	L/D.	$L/\rho c Q^2$.	$D/\rho c Q^2$.	L/D.	$L/\rho c Q^2$.	$D/\rho c Q^2$.	L/D.	$L/\rho c Q^2$.	$D/\rho c Q^2$.	L/D.
°												
15	4·182	1·229	3·41	4·100	1·016	4·03	3·970	0·690	5·76	3·110	0·365	8·53
10	3·523	0·781	4·51	3·369	0·628	5·36	3·286	0·462	7·12	2·509	0·263	9·54
5	2·476	0·378	6·55	2·419	0·343	7·05	2·315	0·259	8·94	1·891	0·178	10·62
1	1·544	0·154	10·00	1·574	0·152	10·36	1·551	0·148	10·50	1·462	0·130	11·25
0	1·300	0·114	11·41	1·286	0·112	11·46	1·285	0·113	11·45	1·282	0·112	11·45

The results contained in this last Table are also given graphically as curves where the lift and drift are plotted against $\beta - \alpha$, fig. 11. The curves 1, 2, 3, and 4 represent the lift for $B = 35^\circ, 45^\circ, 60^\circ$, and 75° respectively, while (1), (2), (3), (4) represent the drift.

A curve is also given, in which the length of the front plane is plotted against values of B (fig. 12).

In conclusion we now proceed to examine the above Tables. It is evident that in all cases both the lift and drift are largest when the angle between the planes is largest (for the values of the angle under consideration). This evidently points to the fact that the ratio of lift to the area of the plane is greater, the greater the bend. But the drift is also increased, which is

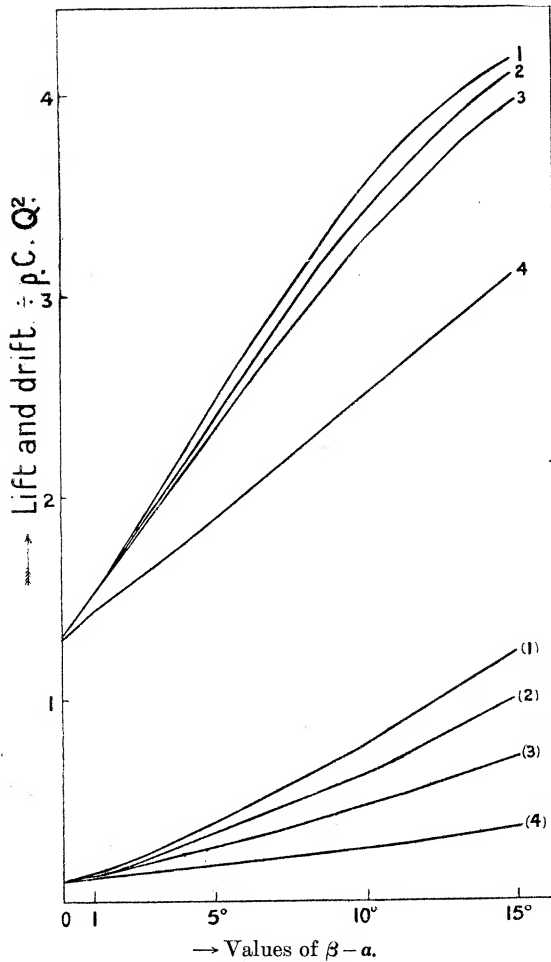


FIG. 11.

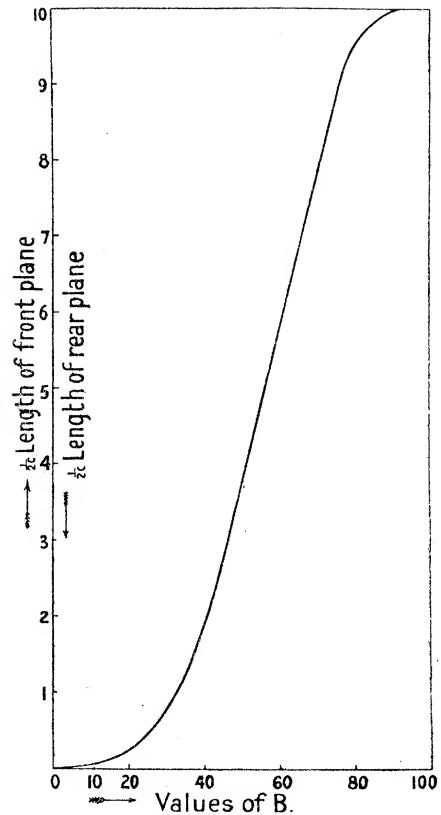


FIG. 12.

not desirable in the case of an aeroplane. Again the ratio of lift to drift is greatest when the angle between the planes is least, but the lift is small.

These two conclusions agree with the experimental results that aerofoils with a large camber give a better ratio of lift to surface, whereas nearly flat aerofoils give a better ratio of lift to drift.

The figures given in Table I show, however, that the ratio of lift to drift, for a large value of m even, is greater than the corresponding ratio for the chord, and the lift on a bent plane is much greater than on its chord. Hence it appears that a bent plane is decidedly more efficient than its chord.

Further, from the last Table it is seen that as the length of the front plane increases, the total lift decreases, whereas the ratio of lift to drift increases.

Hence a plane bent near the leading edge gives a better ratio of lift to surface, while on the other hand a plane bent near the rear edge gives a better ratio of lift to drift. Combining these last conclusions with the corresponding ones above, a plane with a large bend close to the front edge gives the best ratio of lift to surface, while a fairly flat plane bent near the rear edge gives the best ratio of lift to drift.

All these conclusions, then, tend to show that a distinct advantage is gained by making use of a cambered plane.

The above methods can be applied to find the thrusts on any bent plane if the ratio of the lengths of its front and rear parts be given. Fig. 12 gives immediately the corresponding value of B . Having been given the angle of attack as well, and the angle at which the plane is bent, P and m are easily found. The case of a plane bent twice can also be considered (as shown in § 3) but the integrals become complicated.

The following application of fig. 12 may, in certain cases, be of use. A glance at fig. 12 shows that between certain limits (viz., between 37° and 75° approximately) the length l of the front plane is a linear function of $B = aB + b$, say. If, then, the values of B determining the length of the middle part of the plane lie between these limits, we may proceed thus.

Equation 8, § 2, gives in this case

$$z = 2 \int_{B_1}^{B_2} \sin^2(U + P) \sin 2U \left\{ \frac{\sin(U + B_1)}{\sin(U - B_1)} \right\}^{1/2\nu_1} \left\{ \frac{\sin(B_2 + U)}{\sin(B_2 - U)} \right\}^{1/2\nu_2} dU \\ = \left[aU + b \right]_{B_1}^{B_2},$$

where $B_2 > U > B_1$, B_1 and B_2 being the two values of B ,

and ν_1 and ν_2 are the angles of bending at B_1 and B_2 ,

therefore

$$a = 2 \sin^2(U + P) \sin 2U \left\{ \frac{\sin(U + B_1)}{\sin(U - B_1)} \right\}^{1/2\nu_1} \left\{ \frac{\sin(B_2 + U)}{\sin(B_2 - U)} \right\}^{1/2\nu_2},$$

$$\text{or} \quad \left\{ \frac{\sin(U + B_1)}{\sin(U - B_1)} \right\}^{1/2\nu_1} \left\{ \frac{\sin(B_2 + U)}{\sin(B_2 - U)} \right\}^{1/2\nu_2} = \frac{a}{2 \sin^2(U + P) \sin 2U},$$

which, when substituted in the formula corresponding to 10, § 2, gives the pressure difference

$$\frac{4}{a} \int_{B_1}^{B_2} \sin^2(U + P) \sin^2(U - P) \sin^2 2U dU.$$

It should be observed that this is not independent of ν_1 and ν_2 , these being included in P . This integral reduces to

$$\frac{4}{a} \int \sin^2 2U \sin^4 U dU - \frac{8}{a} \sin^2 P \int \sin^2 2U \sin^2 U dU + \frac{4}{a} \sin^4 P \int \sin^2 2U dU,$$

the three integrals being easily reducible.*

* Williamson, 'Int. Calc.,' Ch. III.

This method has been tested and found to give results which agree closely with those obtained by the more laborious method applied to the case of the plane with one bend.

The Difference between the Magnetic Diurnal Variations on Ordinary and Quiet Days at Kew Observatory.

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§ 1. While investigating the nature of the diurnal inequality of the magnetic elements at Kew Observatory, I have discovered a somewhat remarkable relationship between the inequalities derived respectively from quiet days and from ordinary days (*i.e.* all days except those of large disturbance). The period whose records were employed consisted of the 11 years 1890 to 1900. During that period five days were selected for each month by the Astronomer Royal as typical of quiet conditions. An analysis of the diurnal inequalities at Kew on these quiet days was given in an earlier paper.* Use is also made of the diurnal inequalities† derived from 209 disturbed days, being the days of principal disturbance from 1890 to 1900. The new material employed consists of diurnal inequalities based on all the days of the 11 years, excluding the 209 highly disturbed days but including the quiet days. The cost of measuring the curves was defrayed by a grant from the Government Grant Committee.

The diurnal variation in the horizontal components of magnetic force is often advantageously studied by considering the vector which represents the force to whose action the departure from the mean value for the day may be ascribed.

In the figure NS and WE are perpendicular lines, respectively in and perpendicular to the geographical meridian. If OM and PM, the co-ordinates of the point P, are respectively proportional to the departures of the east and north components of force at any given hour from their mean values for the day, OP represents in magnitude and direction the force to which the diurnal variation in the horizontal plane may be ascribed. The successive positions occupied by P throughout the 24 hours constitute what is known as the vector diagram.

* 'Phil. Trans.,' A, vol. 202, p. 335.

† 'Phil. Trans.,' A, vol. 210, p. 210.